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Thermal Fluctuations in Some Random Field Models

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Exact identities are derived for a family of models including (a) a domain wall in a random field Ising model (RFIM), and (b) the random anisotropy XY model in the no-vortex approximation. In particular, the second moment of thermal fluctuations is not affected by frozen randomness. It is checked in a one-dimensional model that higher moments are on the contrary strongly enhanced. Thus, thermal fluctuations are strongly non-Gaussian. This reflects excursions between remote potential wells in the phase space. It is shown exactly that the Imry–Ma argument yields a correct evaluation of the field-induced fluctuations for the one-dimensional model.

KEY WORDS: Ising spins; XY spins; domain walls; quenched disorder; random field; random anisotropy; random potential; non-Gaussian thermal fluctuations; Imry–Ma argument; exact correlation functions; replica trick; random walks.

1. GENERAL MODELS CONSIDERED HERE

We consider models defined by the following Hamiltonian:

$$\mathscr{H} = \frac{1}{4} \sum_{ij} J_{ij}(z_i - z_j)^2 + \sum_i V_i(z_i) \tag{1}$$

where z_i are classical (commuting) continuous variables defined at the site i of a d-dimensional (e.g., cubic) Bravais lattice. The J_{ij} are fixed coefficients, which have the symmetry of the lattice, e.g., $J_{ij} = J$ if i and j are neighbors, otherwise $J_{ij} = 0$. Finally the V_i are independent random potentials. According to the choice of V_i, (1) can represent a whole series of systems.

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(a) A first possibility is

$$V_i(z) = A \cos(z - \varphi_i) \tag{2}$$

where φ_i $(0 < \varphi_i < 2\pi)$ is a random parameter with a uniform probability. Then (1) can represent an incommensurate charge density wave system with impurities in the approximation where vortices or vortex lines are ignored. This model was introduced by Efetov and Larkin,[1] who gave an iteration solution, which would be exact if the iteration method were not questionable.[2,12]

(b) Another possible choice[4] describes a Bloch wall in a $(d+1)$-dimensional ferromagnet with weak bond randomness (random bond Ising model, RBIM). Here z_i is the height of the wall above or below site i. The wall is assumed to have no overhang. As stated above, z_i is treated as a continuous variable. However, in order to define V_i, it is appropriate to introduce the interatomic distance a. Then, if $na < z < (n+1)a$, $V_i(z) = V_{in}$, where the V_{in} are independent random variables with, for instance, a Gaussian distribution and

$$\overline{V_{in} V_{jm}} = V_0^2 \delta_{ij} \delta_{nm} \tag{3}$$

Places with $V_{in} < 0$ correspond to weaker bonds and attract the wall. The model makes sense only below the transition temperature T_c and above the lower critical dimension defined by $D = 5/3$.[19]

(c) Another application we want to give of model (1) is a domain wall in a random field Ising model (RFIM).[4] Again z_i is the height of the domain wall in the $(d+1)$th direction, but now

$$V_i(z_i) = \sum_{n=-N''/a}^{N''/a} H_{in} S_{in} \tag{4}$$

where N'' is the size of the system in the $(d+1)$th dimension and S_{in} is the spin at site (i, n):

$$\begin{aligned} S_{in} &= -1 \qquad \text{if} \quad n \leqslant z_i/a \\ S_{in} &= 1 \qquad \text{if} \quad n > z_i/a \end{aligned} \tag{5}$$

The independent random fields H_{in} have, e.g., a Gaussian distribution and

$$\overline{H_{in} H_{jm}} = a H_0^2 \delta_{ij} \delta_{nm} \tag{6}$$

Again the model makes sense only below T_c and above the critical dimension, which is believed to be $d_{cl} + 1 = D_{cl} = 2$.[4–6] The factor a in (6)

[4] See Ref. 3 for a review of interface wandering.

has been introduced in order to ensure the possibility of considering the continuum limit $a \to 0$, $H_0 = \text{const}$.

(d) Another possibility is Nattermann's[7] "random rod model," which has the advantage of being exactly solvable. It is now assumed that the H_{in} in (4) depend only on i, not on n. Then

$$V_i(z_i) = K_i a \sum_{n=-L/a}^{L/a} S_{in} = K_i z_i \tag{7}$$

where the independent random variables K_i satisfy $\overline{K_i K_j} = \delta_{ij} K_0^2$.

In Sections 2 and 3 general identities will be derived. In the following sections they will be applied to system (c), the RFIM. In Section 4 a one-dimensional model is studied and lower bounds are derived for the moments of thermal fluctuations, showing the non-Gaussian character of these fluctuations. In Section 5 the D-dimensional RFIM is treated approximately. In Section 6 the asymptotic behavior of the field-induced fluctuations is determined for the one-dimensional model, extending a zero-temperature treatment[10] and showing the validity of the Imry–Ma argument at low T.

2. THE IDENTITIES

We now derive the identities that constitute the essential result of this paper. They are based on the partition function Z_λ associated with the modified Hamiltonian

$$\mathscr{H}' = \mathscr{H} - T \sum \lambda_i z_i$$

The mean square thermal fluctuations are related to Z_λ by the formula

$$\overline{\langle z_i z_j \rangle} - \overline{\langle z_i \rangle \langle z_j \rangle} = \left(\frac{\partial^2}{\partial \lambda_i \, \partial \lambda_j} \overline{\ln Z_\lambda} \right)_{\lambda = 0} \tag{8}$$

where bars and brackets denote the ensemble average and the thermal average, respectively. More generally, all derivatives of $\overline{\ln Z_\lambda}$ are correlation functions. For instance,

$$(\partial^3 \overline{\ln Z_\lambda} / \partial \lambda_i \, \partial \lambda_j \, \partial \lambda_l)_{\lambda = 0} = \overline{\langle (z_i - \langle z_i \rangle)(z_j - \langle z_j \rangle)(z_l - \langle z_l \rangle) \rangle} \tag{9}$$

$$\begin{aligned}
&(\partial^4 \overline{\ln Z_\lambda} / \partial \lambda_i \, \partial \lambda_j \, \partial \lambda_l \, \partial \lambda_m)_{\lambda = 0} \\
&\quad = \overline{\langle (z_i - \langle z_i \rangle)(z_j - \langle z_j \rangle)(z_l - \langle z_l \rangle)(z_m - \langle z_m \rangle) \rangle} \\
&\qquad - \overline{\langle (z_i - \langle z_i \rangle)(z_j - \langle z_j \rangle) \rangle \langle (z_l - \langle z_l \rangle)(z_m - \langle z_m \rangle) \rangle} - 2\circlearrowleft
\end{aligned} \tag{10}$$

where "$2\circlearrowleft$" means "two permutations."

As will be seen, $\overline{\ln Z_\lambda}$ is a quadratic function of the λ's, so that all derivatives vanish beyond second order and an infinity of combinations of the correlation functions [in particular (9) and (10)] vanish. To see that, we write $\mathscr{H}'$, using (1), as

$$\mathscr{H}' = \tfrac{1}{2} \sum_k J_{0k} \, |z_k - T\lambda_k/J_{0k}|^2 + \sum_i V(z_i) - \tfrac{1}{2} T^2 \sum_k |\lambda_k|^2/J_{0k} \tag{11}$$

where z_k and λ_k are the Fourier transforms

$$z_k = \frac{1}{\sqrt{N}} \sum_i z_i \exp(i\mathbf{k}_i \cdot \mathbf{R}_i)$$

$$\lambda_k = \frac{1}{\sqrt{N}} \sum_i \lambda_i \exp(i\mathbf{k}_i \cdot \mathbf{R}_i)$$

$$J_{0k} = \sum_j J_{ij} \{1 - \exp[i\mathbf{k}_i \cdot (\mathbf{R}_i - \mathbf{R}_j)]\}$$

where N is the number of sites i, j. If one defines $\varphi_k = T\lambda_k/J_{0k}$, $\zeta_k = z_k - \varphi_k$, and $U(\zeta_i) = V(\zeta_i + \varphi_i)$, one can write (11) as

$$\mathscr{H}' = \tfrac{1}{2} \sum_{ij} J_{ij} (\zeta_i - \zeta_j)^2 + \sum_i U(\zeta_i) - \tfrac{1}{2} T^2 \sum_k |\lambda_k|^2/J_{0k} \tag{12}$$

The partition function that corresponds to (12) and to a given distribution of the V's is

$$Z_\lambda(\{V\}) = \exp\left(\tfrac{1}{2} T \sum_k |\lambda_k|^2/J_{0k}\right) Z(\{U\})$$

where Z is the partition function corresponding to $\lambda = 0$ and to the potentials $U(\zeta_i) = V(\zeta_i + \varphi_i)$. Let the Efetov–Larkin model (2) be considered first. There all variables α_i and z_i are continuous and the random fields U and V have the same probability distribution, so that the average value of $\ln Z(\{U\})$ is the same as that of $\ln Z(\{V\})$, i.e., independent of the λ's. We have

$$\overline{\ln Z_\lambda} = \tfrac{1}{2} T \sum_k |\lambda_k|^2/J_{0k} + \overline{\ln Z} \tag{13}$$

All derivatives of (13) of order 3 or higher with respect to the λ's vanish, in particular (9) and (10). The second derivative yields, according to (8), the second moment of thermal fluctuations,

$$\overline{\langle |z_k - \langle z_k \rangle|^2 \rangle} = T/J_{0k} \tag{14}$$

The long-wavelength limit $k \approx 0$ is particularly interesting, namely

$$\overline{\langle |z_k - \langle z_k \rangle|^2 \rangle} = T/Jk^2 \tag{15}$$

The Boltzmann constant is taken equal to 1. Relations (13)–(15) also apply to the continuum version of the Ising model defined by (5) in the limit $a \to 0$ (or for small $k \approx 0$), provided the domain walls are roughened by the random field. This occurs for a space dimension $D = d + 1 < D_{\rm CR} = 5$.[5,6] In the proper lattice version of the Ising model, $D_{\rm CR} = 3$.[8,9] Summary for three-dimensional people: relations (13)–(15), and the consequences for higher correlation functions, apply to the three-dimensional Ising model for small k, and to the Efetov–Larkin model (2) for any k.

In the case of the RBIM (model b of Section 1), $D_c = 5/3$ and $D_{\rm CR} = 5$,[3] so that (13) and (14) are expected to hold for $1 < D < 5$.

The theorem concerning the higher order correlation functions can be stated in the following way, using the expression of cumulants in terms of the moments $\langle z_i z_j z_l \cdots \rangle$: the mean value (averaged on the random fields) of these "thermal" cumulants should be zero except for the second order cumulants. All averaged odd moments and cumulants vanish for symmetry reasons. Even cumulants start with

$$\overline{\langle z_{i_1} z_{i_2} \cdots z_{i_{2p}} \rangle} - \overline{\langle z_{i_1} z_{i_2} \cdots z_{i_{2p-1}} \rangle \langle z_{i_{2p}} \rangle} - (p-1)\circlearrowleft + \cdots = 0 \tag{16}$$

These relations do *not* imply a gaussian distribution of the z's since for instance in the second term of (16), $\overline{\langle \cdots \rangle \langle \cdots \rangle}$ is not equal to $\overline{\langle \cdots \rangle}\,\overline{\langle \cdots \rangle}$. It will be seen in Section 4 that thermal fluctuations $(z_i - \langle z_i \rangle)$ have much broader wings than a gaussian.

Relation (15) has already been obtained by Dotsenko and Feigelman[15] together with other equalities which we do not believe to be correct. Namely, in Ref. 15, we agree with (2.14), but not with (2.13).[2,11,12] Also, relation (15) may be regarded as a special case of more general relations[16,17] valid for models where the interaction term is not harmonic.

3. IDENTITIES IN THE REPLICA LANGUAGE

This section reproduces the results of the preceding one in the language of replicas.

The average of ln Z corresponding to (1) may be written as

$$\overline{\ln Z} = \lim_{n \to 0} (\overline{Z^n} - 1)$$

and

$$\overline{Z^n} = \int DV\, P(\{V\}) \exp\left[-\beta \sum_{\alpha=1}^{n} \mathscr{H}(\{z_{i\alpha}\}) \right] \tag{17}$$

where the index α refers to n independent replicas.

The replica trick is not very useful for the Efetov–Larkin model (2), but can be applied to formula (4) because it is a linear function of the random field. Assuming a Gaussian distribution of the independent random variables H_{in}, one can integrate those variables and obtain

$$\overline{Z^n} = \int Dz \exp(-\beta \tilde{\mathscr{H}}) \tag{18}$$

where

$$\tilde{\mathscr{H}} = \tfrac{1}{4} \sum_{ij\alpha} J_{ij}(z_i^\alpha - z_j^\alpha)^2 + \sum_{i\alpha\gamma} W(z_i^\alpha - z_i^\gamma) \tag{19}$$

and, for $|z| \gg a$,

$$W(z) = (H_0^2/T)\, |z| \tag{20}$$

Under the effect of length rescaling, for instance, this interaction may be expected to renormalize into an analytic function, possibly tractable by perturbation theory. For example,

$$W(z) = (H_0^2/T)(z^2 + a^2)^{1/2} \tag{21}$$

Model (b) of Section 1 can also be put into the replica form. The function $W(z)$ is zero except for $|z| < a$, and may be approximated by a Gaussian

$$W(z) = -(V^2/T) \exp(-z^2/a^2) \tag{22}$$

The random rod model is somewhat different. Equation (19) should be replaced according to (7) by

$$\tilde{\mathscr{H}} = \frac{1}{4} \sum_{ij\alpha} J_{ij}(z_i^\alpha - z_j^\alpha)^2 - \frac{K_0^2}{2T} n^2 \sum_{i\alpha} z_i^{\alpha 2} + \sum_{i\alpha\gamma} W(z_i^\alpha - z_i^\gamma) \tag{23}$$

where

$$W(z) = (K_0^2/4T)\, z^2 \tag{24}$$

Now the calculation proceeds as in Section 2. We add to (19) a term

$$\delta\tilde{\mathscr{H}} = -T \sum \lambda_i z_i^\alpha \tag{25}$$

The transformation

$$\zeta_k^\alpha = z_k^\alpha - T\lambda k/J_{0k}$$

yields

$$\tilde{Z}_\lambda = \exp\left(n \sum_k |\lambda_k|^2 T/2J_{0k}\right) \tilde{Z}_{\lambda=0} \tag{26}$$

Therefore, all derivatives of $\ln \tilde{Z}_\lambda$ with respect to λ_k vanish beyond second order. This can be translated into equalities for the correlation functions. For instance, using (10), one finds

$$\sum_{\alpha\gamma\lambda\mu} (\langle z_i^\alpha z_j^\gamma z_l^\lambda z_m^\mu \rangle - \langle z_i^\alpha z_j^\gamma \rangle \langle z_l^\lambda z_m^\mu \rangle - 2\circlearrowleft) = 0 \tag{27}$$

Second-order derivatives yield

$$\sum_{\alpha\gamma} \langle z_k^\alpha z_{-k}^\gamma \rangle = \frac{nT}{J_{0k}} \simeq \frac{nT}{Jk^2} \tag{28}$$

where the approximate expression holds for small k. Insertion into (27) yields

$$\sum_{\alpha\gamma\lambda\mu} \langle z_k^\alpha z_{-k}^\gamma z_{k'}^\lambda z_{-k'}^\mu \rangle = \left(\frac{T}{Jk^2}\right)^2 (1 + \delta_{kk'})$$

Equality (27) and analogous equalities valid for higher moments are satisfied for a Gaussian distribution of the variables

$$y_i = \sum_{\alpha=1}^{n} z_i^\alpha \tag{29}$$

It can be checked that the equalities in the replica language are the same as those derived in Section 2. One can escape the replica space by the following formulas (derived by averaging the z's *before* random fields):

$$\sum_{\alpha\gamma} \langle z_i^\alpha z_j^\gamma \rangle_{n\to 0} = n\overline{\langle z_i z_j \rangle} + n(n-1)\overline{\langle z_i \rangle \langle z_j \rangle}$$

$$\begin{aligned}\sum_{\alpha\gamma\lambda\mu} \langle z_i^\alpha z_j^\gamma z_l^\lambda z_m^\mu \rangle_{n\to 0} = {} & n\overline{\langle z_i z_j z_l z_m \rangle} + n(n-1)[\overline{\langle z_i z_j \rangle \langle z_l z_m \rangle} + 2\circlearrowleft] \\ & + n(n-1)[\overline{\langle z_i z_j z_l \rangle \langle z_m \rangle} + 3\circlearrowleft] \\ & + n(n-1)(n-2)[\overline{\langle z_i z_j \rangle \langle z_l \rangle \langle z_m \rangle} + 5\circlearrowleft] \\ & + n(n-1)(n-2)(n-3)\overline{\langle z_i \rangle \langle z_j \rangle \langle z_l \rangle \langle z_m \rangle} \\ & \vdots \end{aligned}$$

Equality (28) is satisfied if

$$\langle z_k^\alpha z_{-k}^\gamma \rangle_n = \frac{T}{J_{0k} + n\Delta_k}\,\delta_{\alpha\gamma} + \frac{T\Delta_k}{J_{0k}(J_{0k} + n\Delta_k)} \tag{30}$$

and conversely it can be shown that (30) is a consequence of (28). Here Δ_k is an unknown quantity, except in the random rod model, in which

$$T\Delta_k = K_0^2/4 \tag{31}$$

is independent of k. Since $J_{0k} \approx Jk^2$ for small k, *the second term of* (30) *dominates the first one in the limit* $n \to 0$, $k \to 0$. This qualitative result is probably true for the RFIM as well, as will be seen in the next section. Thus, the second moment (28) of y_k is remarkably *small*. The factors n and T are expected, but it was not expected that the k dependence is characterized by a more weakly divergent factor $1/k^2$, rather than by $T\Delta_k/k^4$ as in (30).

Since the anharmonic perturbation in (19) is a function of $z_i^\alpha - z_i^\gamma$, it is of interest to consider the fluctuations of $z_k^\alpha - z_k^\gamma$. The second moment is, according to (30),

$$\langle |z_k^\alpha - z_k^\gamma|^2 \rangle_n = \frac{2T}{J_{0k} + n\Delta_k} \tag{32}$$

This is again small for $n = 0$. But higher moments are presumably not small. For instance,

$$\begin{aligned} &\langle (z_i^\alpha - z_i^\gamma)(z_j^\alpha - z_j^\gamma)(z_l^\alpha - z_l^\gamma)(z_m^\alpha - z_m^\gamma) \rangle_{n=0} \\ &\quad = 2\overline{\langle (z_i - \langle z_i \rangle)(z_j - \langle z_j \rangle)(z_l - \langle z_l \rangle)(z_m - \langle z_m \rangle) \rangle} \\ &\qquad - 4[\overline{\langle (z_i - \langle z_i \rangle)(z_j - \langle z_j \rangle) \rangle \langle z_l z_m \rangle} + 2\circlearrowleft] \end{aligned} \tag{33}$$

The second term on the right-hand side is large because of the factor $\langle z_l z_m \rangle$, which does not vanish even at $T = 0$. Thus, either the left-hand side of (33), the first term of the right-hand side, or possibly both are also large. This suggests that the second term in (19) probably cannot be treated by perturbation theory.

4. A TOY MODEL

A one-dimensional model will now be treated, where some exact results can be derived.[10,11] It will be seen that the non-Gaussian distribution of thermal fluctuations is related to metastable states. Our toy model is characterized by a single real variable z and a Hamiltonian

$$\mathscr{H} = \tfrac{1}{2} g z^2 + \sum_{p < z/a} H_p \tag{34}$$

where the independent random variables H_p satisfy

$$\overline{H_p H_{p'}} = \delta_{pp'} H_0^2 a \tag{35}$$

The model is analogous to that defined by (4)–(6) if one adds a term $\frac{1}{2} g \sum z_i^2$ in (1). The dimension is $d = 0$ or $D = 1$. The term $gz^2/2$ does not modify the calculations of Sections 2 and 3 except that J_{0k} should be replaced by g, and all indices k, i, j, etc., of z disappear. For instance, (14) should be replaced by

$$\overline{\langle (z - \langle z \rangle)^2 \rangle} = T/g \tag{36}$$

and (10) yields

$$\overline{\langle (z - \langle z \rangle)^4 \rangle} = 3\, \overline{\langle (z - \langle z \rangle)^2 \rangle^2} \tag{37}$$

The high-temperature region, where the fluctuation $\overline{\langle z^2 \rangle}$ is of order T/g, may be handled by perturbation theory. We are interested here in the low-temperature region. According to an argument of Imry and Ma,[4] $\overline{\langle z^2 \rangle}$ is such that the mean square of both terms of (34) have the same order of magnitude: $g\overline{\langle z^2 \rangle} \approx H_0 \overline{\langle z^2 \rangle}^{1/4}$, or

$$\overline{\langle z^2 \rangle} \approx (H_0/g)^{4/3} \tag{38}$$

The temperature is assumed to be so low that (38) is much larger than (36). Also, (38) must be much larger than a, otherwise the continuum approximation could not be used.

The result (38) can actually be derived exactly[10,11] at least at $T = 0$, since a lower and an upper bound of $\overline{\langle z^2 \rangle}$ can be obtained, which both are equal to $(H_0/g)^{4/3}$ times a numerical constant. The derivation is given in Section 6, extending a previous calculation[10] to nonvanishing temperatures. Relation (38) is actually still controversial. For instance, Minchau and Pelcovits[18] give a Bogoljubov inequality which violates (38). [We believe that their inequality is only correct when the number n of replicas is larger than 1. Otherwise they are not entitled to go from their relation (16) to their relation (17).]

Thermal fluctuations can now be estimated. Let the minimum E of (34) for $z > 0$ correspond to $z = z_1$ and the minimum E_2 for $z < 0$ correspond to

$z = z_2$. If $|E_1 - E_2| < T$, the thermal fluctuation $\langle \delta z^2 \rangle$ is of order $(z_1 - z_2)^2$, which is of order $(H_0/g)^{4/3}$ according to (38). On the other hand, the typical order of magnitude of E_1 and E_2 is $-H_0(H_0/g)^{1/3}$, as can be seen from the Imry–Ma argument[4] or proved exactly.[10] Thus, the probability that $|E_1 - E_2| < T$ is of order

$$P_T \approx (T/H_0)(H_0/g)^{-1/3} \tag{39}$$

If one multiplies P_T by $(z_1 - z_2)^2$, one finds something of order T/g, the exact second moment (36). This means that large thermal fluctuations, $z - \langle z \rangle \approx \langle z^2 \rangle^{1/2}$, yield a finite contribution to the second moment. Of course weaker fluctuations (e.g., corresponding to two relative minima z_1, z_1' with identical sign) also contribute (Fig. 1).

The above argument yields upper bounds for higher moments as well. We have

$$\overline{\langle (z - \langle z \rangle)^{2p} \rangle} \gtrsim P_T \overline{(x_1 - x_2)^{2p}} \approx (T/H_0)(H_0/g)^{(4p-1)/3} \tag{40}$$

At low temperature this is much larger than $[\overline{\langle (z - \langle z \rangle)^2 \rangle}]^p$. This reflects the strongly non-Gaussian shape of thermal fluctuations.

5. DOMAIN WALLS IN THE *D*-DIMENSIONAL RFIM ($2 \leqslant D = d + 1 < 5$)

The most natural method to treat the RFIM is the renormalization group. Rescaling the lengths, i.e., dividing N by b^{ld}, would hopefully con-

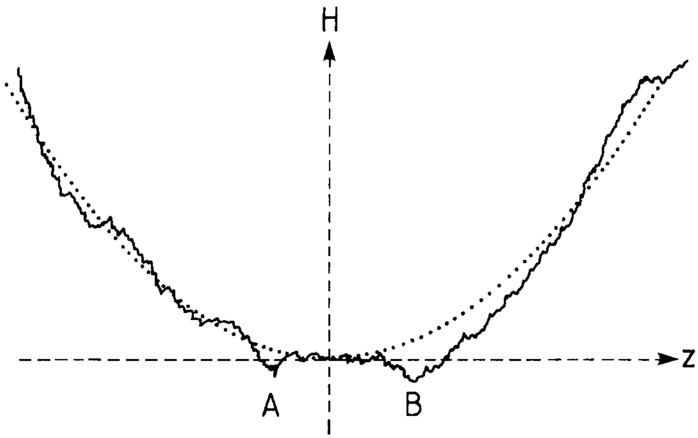

Fig. 1. The Hamiltonian (34). At finite temperature the probability that a secondary minimum B exists, with an energy difference $\mathscr{H}(B) - \mathscr{H}(A) < k_B T$, but far from the absolute minimum A, cannot be ignored. Its contribution to higher moments is large.

serve the forms (1) and (4)–(6), at least with a good approximation. Of course the parameters g, H_0, and a are modified. However, this program was never carried out in a satisfactory way. The simplest approximation, possibly exact,[12,13] is the following.[5,6]

1. J should be replaced by $J_l = Jb^{(d-2)l}$ as in the absence of random field.
2. H_0^2 should be replaced by the mean square random field in a slice of volume b^{ld}

$$H_0 \rightarrow H_l = H_0 b^{ld/2}$$

3. The renormalization of a is probably not crucial for static properties.

An alternative picture is a mere adaptation of the Imry–Ma argument.[14] The domain wall is argued to consist of bumps of area b^{ld}, the heights z of which are the degrees of freedom of the system. This picture is simpler, but equivalent to the renormalization group picture if one neglects the interaction between bumps. Then each bump is represented by Hamiltonian (34) with $g = J_l$ and H_0 replaced by H_l. Formula (38) then yields the well-known result[14]

$$\langle z^2 \rangle \approx (H_0/J)^{4/3} \, b^{2(4-d)l/3} \tag{41}$$

for the mean square height of a bump of linear size b^l. This implies that the function $T\Delta_k$ in (30) is of order

$$T\Delta_k \approx H_0^{4/3} J^{2/3} k^{(5-D)/3} \tag{42}$$

This can be checked by inserting (42) into (30) and noting that integration over $k > b^{-l}$ in the $(D-1)$-dimensional space yields (41), with $d = D - 1$.

Formula (40) provides an evaluation of higher order correlation functions. For instance, if r, r', r'', $r''' \approx b^l$, then

$$\overline{\langle (\delta z_{\mathbf{i}+\mathbf{r}} - \delta z_i)(\delta z_{\mathbf{i}+\mathbf{r}'} - \delta z_i)(\delta z_{\mathbf{i}+\mathbf{r}''} - \delta z_i)(\delta z_{\mathbf{i}+\mathbf{r}'''} - \delta z_i) \rangle}$$
$$\approx (T/H_l)(H_l/g_l)^{7/3}$$

One can wonder if the identities of Sections 2 and 3 might become wrong because of the strong fluctuations (the effect known in spin glasses as replica symmetry breaking). The answer is no. At $T = 0$ the system has a nondegenerate ground state with probability 1. At $T \neq 0$ a few degrees of freedom become "active," but the probability $P_T(l)$ that a bump of size b^l becomes "active" is small at low T according to (39):

$$P_T(l) \approx (T/H_l)(H_l/J_l)^{-1/3} \approx (T/H_0)(H_0/J)^{-1/3}\, b^{-(2+d)l/3}$$

Thus, most of the points of the Bloch wall have a well-defined position at low T and are weakly affected by thermal fluctuations, although the second moment $\overline{\langle z_i^2 \rangle} - \overline{\langle z_i \rangle^2}$ diverges for large sizes according to (15).

6. UPPER AND LOWER BOUND FOR $\overline{\langle z^2 \rangle}$ AT FINITE TEMPERATURE

In this section bounds for the *finite*-temperature correlation

$$\overline{\langle z^2 \rangle} = \overline{[\mathrm{Tr}(e^{-\beta \mathscr{H}(z)} z^2)/\mathrm{Tr}(e^{-\beta \mathscr{H}(z)})]}$$

are evaluated (the bar indicates the configurational average over the random fields). This section is organized as follows: (1) notations, (2) upper bound for $\overline{\langle z^2 \rangle}$, (3) lower bound for $\overline{\langle z^2 \rangle}$, and (4) upper bound for

$$\mathrm{Prob}\left\{ \min_{0 \leqslant l < L} \sum_{\mu=0}^{l} X_\mu \leqslant -m \right\}$$

where the latter is the probability that the minimum of all symmetric random walks of length L starting at $\sum_{\mu=0}^{0} X_\mu = 0$ is dominated by $-m$, $m \geqslant 0$.

The existence of those bounds for $\overline{\langle z^2 \rangle}$ implies

$$\overline{\langle z^2 \rangle} \sim (H_0/g)^{4/3}$$

which would also result from a straightforward argument à la Imry and Ma.[4]

6.1. Notation

The discretized version of the toy model of Section 4 reads ($z = l \cdot a$, $l = 0, \pm 1, \pm 2, \ldots$)

$$\mathscr{H}(l) = \tfrac{1}{2} g (a \cdot l)^2 + H_0 a^{1/2} \sum_{\mu=-\infty}^{l} X_\mu \tag{43}$$

where g is the strength of the harmonic potential >0; H_0 is the strength of the random field; X_μ denotes Ising-type variables with values ± 1, and a is the lattice constant. Introducing the parameter

$$\eta = \tfrac{1}{2}(g/H_0)\, a^{3/2} \tag{44}$$

we find that the Hamiltonian becomes for positive l

$$\mathscr{H}(l) = H_0 a^{1/2}\left(\eta l^2 + \sum_{\mu=0}^{l} X_\mu\right) \tag{45}$$

The summation in (45) is restricted to $\mu \geqslant 0$, since the term with $\sum_{\mu=-\infty}^{-1} X_\mu$ drops out in the formula of $\overline{\langle z^2 \rangle}$.

6.2. Upper Bound for $\overline{\langle z^2 \rangle}$

The correlation for $l(z = l \cdot a,\ l = 0,\ \pm 1,\ \pm 2, \ldots)$ reads

$$\overline{\langle l^2 \rangle} = \overline{\left[\left(\sum_{l=-\infty}^{+\infty} e^{-\beta \mathscr{H}(l)} l^2\right)\Big/\left(\sum_{l=-\infty}^{+\infty} e^{-\beta \mathscr{H}(l)}\right)\right]} \tag{46}$$

Omit the summation over negative and positive integers, respectively, in the denominator and notice that, since the X_μ are identically distributed, the random walks $\sum_{\mu=0}^{l} X_\mu$ and $\sum_{\mu=0}^{-l} X_\mu$ have the same distribution:

$$\overline{\langle l^2 \rangle} \leqslant 2\overline{\left[\left(\sum_{l=0}^{\infty} e^{-\beta \mathscr{H}(l)} l^2\right)\Big/\left(\sum_{l=0}^{\infty} e^{-\beta \mathscr{H}(l)}\right)\right]} \tag{47}$$

Now introduce an arbitrary length scale $L \gg 1$ and reorganize the summations

$$\sum_{l=0}^{\infty} = \sum_{m=0}^{\infty} \sum_{l=mL}^{(m+1)L-1}$$

Setting

$$Z_m = \sum_{l=mL}^{(m+1)L-1} e^{-\beta \mathscr{H}(l)} \tag{48}$$

taking the upper bound for l^2 on each interval $[m \cdot L, (m+1) \cdot L]$, and isolating the first term of the sum in the nominator, we arrive for a frozen configuration of random fields at

$$\langle l^2 \rangle \leqslant 2L^2 \left\{\frac{Z_0}{\sum_{m=0}^{\infty} Z_m} + \sum_{m=1}^{\infty} (m+1)^2 \frac{Z_m}{\sum_{n=0}^{\infty} Z_n}\right\} \tag{49}$$

The rhs of (49) grows if certain terms of the sums in the denominators are left out,

$$\langle l^2 \rangle \leqslant 2L^2 \left\{1 + \sum_{m=1}^{\infty} (m+1)^2 \frac{Z_m}{Z_0 + Z_m}\right\} \tag{50}$$

In the following it is shown that for a special choice of L the sum in (50) is dominated by a finite numerical constant and the desired scaling $\overline{\langle z^2 \rangle} \sim (H_0/g)^{4/3}$ is achieved.

To this end, define

$$\varepsilon_m = \min_{mL \leqslant l < (m+1)L} \mathscr{H}(l), \qquad E_0 = \max_{0 \leqslant l < L} \mathscr{H}(l) \tag{51}$$

(which are still random variables depending on $\sum_{\mu=0}^{l} X_\mu$).

For every configuration of the random field we have the following alternative (see Fig. 2):

$$\begin{aligned} &1. \quad \varepsilon_m > E_0: \qquad Z_m/(Z_0 + Z_m) \leqslant e^{-\beta(\varepsilon_m - E_0)} \\ &2. \quad \varepsilon_m \leqslant E_0: \qquad Z_m/(Z_0 + Z_m) \leqslant 1 \end{aligned} \tag{52}$$

and we have to calculate the corresponding probabilities, i.e., the weights of the related fluctuations of $\mathscr{H}(l)$ between the intervals $[0, L]$ and $[mL, (m+1)L]$.

Averaging over the random field gives

$$\overline{\langle l^2 \rangle} \leqslant 2L^2 \left\{ 1 + \sum_{m=1}^{\infty} (m+1)^2 (X_m + Y_m) \right\} \tag{53}$$

referring to case 1,

$$X_m = \int_0^\infty dE_0\, \rho(E_0) \int_0^\infty dE\, \mathscr{P}_m(E, E_0)\, e^{-\beta E} \tag{54}$$

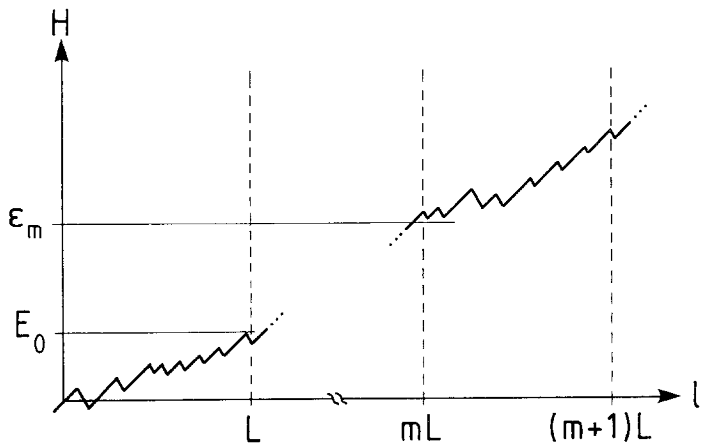

Fig. 2. Realization of $\mathscr{H}(l)$ with $\varepsilon_m > E_0$ (case 1; see text).

and case 2,

$$Y_m = \int_0^\infty dE\, \rho(E_0) \int_{-\infty}^0 dE\, \mathscr{P}_m(E, E_0)$$
$$= \int_0^\infty dE_0\, \rho(E_0)\, p_m(E_0) \tag{55}$$

where

$$E = \varepsilon_m - E_0 \quad (>0 \text{ in case } 1)$$

$\rho(E_0)$ is the distribution of E_0, and

$$\mathscr{P}_m(E, E_0) = \text{Prob}\{E \leqslant \varepsilon_m - E_0 \leqslant E + dE,\ E_0 \text{ fixed}\}/dE$$
$$p_m(E_0) = \text{Prob}\{\varepsilon_m \leqslant E_0,\ E_0 \text{ fixed}\}$$

Now we proceed to calculate upper bounds for X_m and Y_m separately.

Case 2. Upper bound for Y_m. Split the E_0-integration into $Y_m = Y_m^{(1)} + Y_m^{(2)}$, with

$$Y_m^{(1)} = \int_0^A dE_0\, \rho(E_0)\, p_m(E_0) \tag{56}$$

$$Y_m^{(2)} = \int_A^\infty dE_0\, \rho(E_0)\, p_m(E_0) \tag{57}$$

$$\begin{aligned} A &= \tfrac{1}{2} H_0 a^{1/2} \eta m^2 L^2 \\ \eta &= \tfrac{1}{2}(H_0/g)\, a^{3/2} \end{aligned} \tag{58}$$

Equation (56) reads

$$Y_m^{(1)} = \text{Prob}\{\varepsilon_m < E_0 < A\}$$

and, relaxing the restriction for E_0,

$$Y_m^{(1)} \leqslant \text{Prob}\{\varepsilon_m < A\}$$

The inequality $\varepsilon_m < A$, i.e.,

$$\min_{mL \leqslant l < (m+1)L} \left[H_0 a^{1/2} \left(\eta l^2 + \sum_{\mu=0}^{l} X_\mu \right) \right] < \tfrac{1}{2} H_0 a^{1/2} \eta m^2 L^2$$

implies

$$\min_{mL \leqslant l < (m+1)L} \sum_{\mu=0}^{l} X_\mu \leqslant -\tfrac{1}{2}\eta m^2 L^2$$

and

$$Y_m^{(1)} \leqslant \text{Prob}\left\{\min_{mL \leqslant l < (m+1)L} \sum_{\mu=0}^{l} X_\mu \leqslant -\tfrac{1}{2}\eta m^2 L^2\right\}$$

Using (105) of Section 6.4 results in

$$Y_m^{(1)} \leqslant 2 \exp\left[-\frac{m^4 \eta^2 L^3}{8(m+1)}\right] \tag{59}$$

Since $p_m \leqslant 1$, Eq. (57) implies

$$Y_m^{(2)} \leqslant \text{Prob}\{E_0 \geqslant \tfrac{1}{2} H_0 a^{1/2} \eta m^2 L^2\}$$

i.e., the realizations have to satisfy

$$E_0 = \max_{0 \leqslant l < L} \left[H_0 a^{1/2} \left(\eta l^2 + \sum_{\mu=0}^{l} X_\mu\right)\right] \geqslant \tfrac{1}{2} H_0 a^{1/2} \eta m^2 L^2$$

which implies

$$\max_{0 \leqslant l < L} \sum_{\mu=0}^{l} X_\mu \geqslant \tfrac{1}{2}\eta L^2 (m^2 - 2)$$

A consequence of reflection symmetry of the random walks is

$$\text{Prob}\left\{\min_{0 \leqslant l < L} \sum_{\mu=0}^{l} X_\mu \leqslant -u\right\} = \text{Prob}\left\{\max_{0 \leqslant l < L} \sum_{\mu=0}^{l} X_\mu \geqslant u\right\}$$

so that

$$\begin{aligned} Y_m^{(2)} &\leqslant \text{Prob}\left\{\min_{0 \leqslant l < L} \sum_{\mu=0}^{l} X_\mu \leqslant -\tfrac{1}{2}\eta L^2 (m^2 - 2)\right\} \\ &\leqslant 2 \exp\left[-\frac{\eta^2 L^3}{8} (m^2 - 2)^2\right] \end{aligned} \tag{60}$$

yielding with (59)

$$Y_m \leqslant 2 \left\{\exp\left(-\frac{1}{8} \eta^2 L^3 \frac{m^4}{m+1}\right) + \exp\left[-\frac{1}{8} \eta^2 L^3 (m^2 - 2)^2\right]\right\} \tag{61}$$

Case 1. Upper bound for X_m. Again, split the E_0-integration in (54) into $X_m = X_m^{(1)} + X_m^{(2)}$, with

$$X_m^{(1)} = \int_0^A dE_0\, \rho(E_0) \int_0^\infty dE\, \mathscr{P}_m(E, E_0)\, e^{-\beta E} \tag{62}$$

$$X_m^{(2)} = \int_A^\infty dE_0\, \rho(E_0) \int_0^\infty dE\, \mathscr{P}_m(E, E_0)\, e^{-\beta E} \tag{63}$$

with abbreviations as in (58).

Notice that $e^{-\beta E} \leqslant 1$ and $\int_0^\infty dE\, \mathscr{P}_m(E, E_0) \leqslant 1$, so that (63) implies

$$\begin{aligned} X_m^{(2)} &\leqslant \text{Prob}\left\{ \max_{0 \leqslant l < L} \sum_{\mu=0}^{l} X_\mu \geqslant \tfrac{1}{2}\eta L^2 (m^2 - 2)^2 \right\} \\ &\leqslant 2 \exp[-\tfrac{1}{8}\eta^2 L^3 (m^2 - 2)^2] \end{aligned} \tag{64}$$

with the same reasoning as for $Y_m^{(2)}$.

Now split the E-integration in (62) into $X_m^{(1)} = a_m + b_m$, with

$$a_m = \int_0^A dE_0\, \rho(E_0) \int_0^{A/2} dE\, e^{-\beta E} \mathscr{P}_m(E, E_0) \tag{65}$$

$$b_m = \int_0^A dE_0\, \rho(E_0) \int_{A/2}^\infty dE\, e^{-\beta E} \mathscr{P}_m(E, E_0) \tag{66}$$

Taking the lower bound for $e^{-\beta E}$ on $[A/2, \infty]$ and observing that all remaining probabilities are dominated by 1, we find that Eq. (66) results in

$$b_m \leqslant \exp(-\beta \tfrac{1}{4} H_0 a^{1/2} \eta L^2 m^2) \tag{67}$$

Now to a_m: use $e^{-\beta E} \leqslant 1$

$$\begin{aligned} a_m &\leqslant \int_0^A dE_0\, \rho(E_0)\, \text{Prob}\{0 < E < A/2\} \\ &\leqslant \int_0^A dE_0\, \rho(E_0)\, \text{Prob}\{E_0 < \varepsilon_m < E_0 + A/2\} \end{aligned}$$

i.e.,

$$a_m \leqslant \text{Prob}\{\varepsilon_m \leqslant 3/2A\}$$

Analogously to $Y_n^{(1)}$, one gets

$$a_m \leqslant 2 \exp\left(-\frac{9}{32} \eta^2 L^3 \frac{m^4}{m+1}\right) \tag{68}$$

Equations (64), (67), and (68) imply for X_m,

$$X_m \leqslant 2\exp[-\tfrac{1}{8}\eta^2L^3(m^2-2)^2] + \exp(-\beta\tfrac{1}{4}H_0a^{1/2}\eta L^2m^2) + 2\exp[-\tfrac{9}{32}\eta^2L^3(m^4/(m+1))] \quad (69)$$

Relations (61) and (69) yield an upper bound for the ratio

$$\overline{[Z_m/(Z_0+Z_m)]} \leqslant 4\exp[-\tfrac{1}{8}\eta^2L^3(m^2-2)^2] + 4\exp[-\tfrac{1}{8}\eta^2L^3(m^4/(m+1))] + \exp(-\tfrac{1}{4}\beta H_0a^{1/2}\eta L^2\eta m^2)$$

Using

$$\left.\begin{aligned}(m^2-2)^2 &\geqslant \tfrac{1}{4}m^4\\ m^4/(m+1) &\geqslant \tfrac{1}{2}m^3\end{aligned}\right\} \qquad m \geqslant 1$$

one gets

$$\overline{\langle l^2\rangle} \leqslant 2L^2\left\{1+\sum_{m=1}^{\infty}(m+1)^2\left[4\exp\left(-\frac{m^4}{32}\eta^2L^3\right)+4\exp\left(-\frac{m^3}{16}\eta^2L^3\right) + \exp\left(-\frac{1}{4}\beta H_0a^{1/2}L^2\eta m^2\right)\right]\right\} \quad (70)$$

The choice for the length scale L

$$\eta^2L^3 = 1 \quad (71)$$

implies

$$(aL)^{1/2} = (2H_0/g)^{1/3}$$

$$L^2 = \eta^{-4/3} = (2H_0/g)^{4/3}\,a^{-2} \quad (72)$$

$$\overline{\langle l^2\rangle} \leqslant a^{-2}f(g, H_0, T)(H_0/g)^{4/3} \quad (73)$$

with

$$f(g, H_0, T) = 4\cdot 2^{1/3}\left(1+\sum_{m=1}^{\infty}(m+1)^2\left\{4\exp\left(-\frac{m^4}{32}\right)+4\exp\left(-\frac{m^3}{16}\right) + \exp\left[-\frac{1}{2}m^2\frac{H_0}{T}\left(\frac{H_0}{4g}\right)^{1/3}\right]\right\}\right) \quad (74)$$

The infinite sums in (74) do converge provided $H_0 > 0$ and g, $T < \infty$. The choice of L is also compatible with the former assumption that $L \gg 1$, since $L \sim a^{-1}$. The function f of (74) has the scaling form

$$f(g, H_0, T) = c + \tilde{f}[H_0/T, (H_0/g)^{4/3}] \quad (75)$$

with

$$c = 4 \cdot 2^{1/3} \left\{ 1 + 4 \sum_{m=1}^{\infty} (m+1)^2 \left[\exp\left(-\frac{m^4}{32} \right) + \exp\left(-\frac{m^3}{16} \right) \right] \right\}$$
$$\simeq 474.1 \qquad (76)$$

$$\mathcal{J}(x, y) = 4 \cdot 2^{1/3} \sum_{m=1}^{\infty} (m+1)^2 \exp(-\tfrac{1}{2} m^2 x y^{1/4} \cdot 4^{-1/3})$$

The regime of interest is the one of *strong* random fields $g \ll H_0$ and *low* temperatures. The choice, e.g.,

$$H_0/T > 1, \qquad H_0/4g > 8 \qquad (77)$$

leads to the upper bound for $\mathcal{J}$

$$\mathcal{J} \leqslant 4 \cdot 2^{1/3} \sum_{m=1}^{\infty} (m+1)^2 \exp(-m^2) \simeq 8.3$$

which means

$$\overline{\langle l^2 \rangle} \leqslant (H_0/g)^{4/3} a^{-2} \cdot 482.4$$

subject to the conditions of (77).

6.3. Lower Bound for $\overline{\langle z^2 \rangle}$

Introduce two length scales L_1 and $L_2 L_1$ and define

$$\varepsilon_1 = \min_{|l| < L_1} \mathscr{H}(l), \qquad E_2 = \max_{L_2 \leqslant l < L_2 + L_1} \mathscr{H}(l) \qquad (78)$$

Now consider the probability p_1 that fluctuations of $\mathscr{H}(l)$ between the two intervals $[-L_1, +L_1]$ and $[L_2, L_2 + L_1]$ are at least of size E, i.e. (see also Fig. 3)

$$p_1 = \text{Prob}\{ \min_{|l| < L_1} \mathscr{H}(l) > -E \text{ and } \max_{L_2 \leqslant l < L_2 + L_1} \mathscr{H}(l) < -2E \}$$
$$= p_1(L_1, L_2, E), \qquad E > 0 \qquad (79)$$

With the help of p_1 we can construct a lower bound for $\overline{\langle l^2 \rangle}$ such as

$$\overline{\langle l^2 \rangle} \geqslant \tfrac{1}{3} p_1 L_1^2 \qquad (80)$$

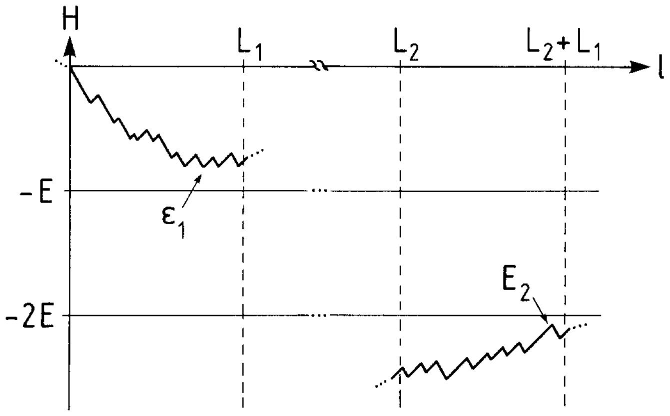

Fig. 3. Realization of $\mathscr{H}(l)$ with $\varepsilon_1 > -E$ and $E_2 < -2E$ (for definitions of ε_1 and E_2 see text).

Condition (80) is reached by dropping the summation over $|l| < L_1$, taking L_1^2 instead of l^2 in the nominator of $\overline{\langle l^2 \rangle}$ [see (46)], and noticing that

$$\begin{aligned} 2 \sum_{|l| \geqslant L_1} e^{-\beta \mathscr{H}(l)} &\geqslant 2 \sum_{l=L_2}^{L_2+L_1} e^{-\beta \mathscr{H}(l)} \\ &\geqslant 2L_1 e^{-\beta E_2} \\ &\geqslant 2L_1 e^{2\beta E} \qquad \text{with probability } p_1 \\ &\geqslant 2L_1 e^{-\beta \varepsilon_1} \\ &\geqslant \sum_{|l| < L_1} e^{-\beta \mathscr{H}(l)} \end{aligned} \tag{81}$$

The remaining task is to show that p_1 is *not* zero for a choice of L_1, L_2, and E that yields the desired scaling behaviour of $\overline{\langle l^2 \rangle}$.

To this end, define

$$A(L; w, u) = \text{Prob}\{\mathscr{H}(L) = -u \text{ and } \min_{0 \leqslant l < L} \mathscr{H}(l) > -w\} \tag{82a}$$

(i.e., $A = 0$ in case of $-u < -w$), and

$$B(L; w) = \text{Prob}\left\{\max_{0 \leqslant l < L} H_0 a^{1/2} \sum_{\mu=0}^{l} X_\mu < w\right\} \tag{82b}$$

$$\begin{aligned} Y(L; u) &= \text{Prob}\left\{H_0 a^{1/2} \sum_{\mu=0}^{l} X_\mu = -u\right\} \\ &= Y(L; -u), \qquad \text{because of reflection symmetry} \end{aligned} \tag{82c}$$

Now look for an event included in p_1, since its probability is a lower bound for p_1. Such an event can be constructed as follows: (i) give a *sufficient* condition to satisfy $E_2 \leqslant -2E$; (ii) impose $\varepsilon_1 > -E$ with fixed $\mathscr{H}(L_1)$; (iii) give the condition that (i) and (ii) can be satisfied simultaneously.

Condition (i) is implied by

$$\mathscr{H}(L_2) = -3E - v, \qquad v > 0$$

together with

$$\max_{0 \leqslant l < L_1} H_0 a^{1/2} \sum_{\mu=0}^{l} X_{L_2+\mu} < E + v - H_0 a^{1/2} \eta [(L_2 + L_1)^2 - L_2^2] \tag{83}$$

The corresponding distribution is $B(L_1; E + v - H_0 a^{1/2} \eta (2L_1 L_2 + L_1^2))$. Condition (ii) reads

$$\min_{0 \leqslant l < L_1} \mathscr{H}(l) > -E \qquad \text{and} \qquad \mathscr{H}(L_1) = -u \tag{84}$$

with distribution $A(L_1; E, u)$. We are left to get from $\mathscr{H}(L_1) = -u$ to $\mathscr{H}(L_2) = -3E - v$. Condition (iii) is: The random walk has to travel the difference $\mathscr{H}(L_2) - \mathscr{H}(L_1) = -3E - v + u$ in $L_2 - L_1$ steps, i.e.,

$$-H_0 a^{1/2} \sum_{\mu=0}^{L_2 - L_1} X_{L_1+\mu} = 3E + v - u + H_0 a^{1/2} \eta (L_2^2 - L_1^2) \tag{85}$$

with distribution $Y(L_2 - L_1; 3E + v - u + H_0 a^{1/2} \eta (L_2^2 - L_1^2))$. Conditions (i)–(iii) imply the event for p_1:

$$\begin{aligned} p_1 \geqslant & \int_{-\infty}^{+\infty} du\, A(L_1; E, u) \\ & \times \int_0^{\infty} dv\, Y(L_2 - L_1; 3E + v - u + H_0 a^{1/2} \eta (L_2^2 - L_1^2)) \\ & \times B(L_1; E + v - H_0 a^{1/2} \eta (2L_1 L_2 + L_1^2)) \end{aligned} \tag{86}$$

Since

$$\min_{0 \leqslant l < L_1} H_0 a^{1/2} \sum_{\mu=0}^{l} X_\mu > -E \qquad \text{and} \qquad H_0 a^{1/2} \sum_{\mu=0}^{L_1} X_\mu = -u - H_0 a^{1/2} \eta L_1^2$$

with distribution $A_0(L_1; E, u + H_0 a^{1/2} \eta L_1^2)$ implies the event

$$\min_{0 \leqslant l < L_1} \mathscr{H}(l) > -E \qquad \text{and} \qquad \mathscr{H}(L_1) = -u$$

we have

$$p_1 \geqslant \int_{-\infty}^{+\infty} du\, A_0(L_1; E, u + H_0 a^{1/2} \eta L_1^2)$$

$$\times \int_0^{\infty} dv\, Y(L_2 - L_1; 3E + v - u - H_0 a^{1/2} \eta L_1^2 + H_0 a^{1/2} \eta L_2^2)$$

$$\times B(L_1; E + v - 2H_0 a^{1/2} \eta L_1 L_2 - H_0 a^{1/2} \eta L_1^2) \tag{87}$$

Set $u' = u + H_0 a^{1/2} \eta L_1^2$ and relax the $-u'$ in Y in inequality (87), which means that the random walk has to travel even a larger distance between L_1 and L_2; one gets

$$p_1 \geqslant \int_0^{+\infty} du\, A_0(L_1; E, u)$$

$$\times \int_0^{\infty} dv\, Y(L_2 - L_1; 3E + v + H_0 a^{1/2} \eta L_2^2)$$

$$\times B(L_1; E + v - 2H_0 a^{1/2} \eta L_1 L_2 - H_0 a^{1/2} \eta L_1^2) \tag{88}$$

Also, the v in B of (88) can be set to zero to get the rhs smaller, i.e., the random walk reaches at most $-2E - v$ in the interval $[L_2; L_2 + L_1]$. Since the random walks start at the origin, the second argument of B is restricted to be positive and should be of the order of $L_1^{1/2}$:

$$E - 2H_0 a^{1/2} \eta L_1 L_2 - H_0 a^{1/2} \eta L_1^2 = \lambda H_0 a^{1/2} L_1^{1/2}, \qquad \lambda > 0 \tag{89}$$

Setting

$$3E = H_0 a^{1/2} \eta L_2^2 \tag{90}$$

we have

$$p_1 \geqslant \int_0^{\infty} du\, A_0(L_1; \tfrac{1}{3} H_0 a^{1/2} \eta L_2^2, u) \int_0^{\infty} dv\, Y(L_2 - L_1; v + 2H_0 a^{1/2} \eta L_2^2)$$

$$\times B(L_1; \tfrac{1}{3} H_0 a^{1/2} \eta L_2^2 - 2H_0 a^{1/2} \eta L_1 L_2 - H_0 a^{1/2} \eta L_1^2) \tag{91}$$

Equation (90) eliminates the parameter E and (89) provides us with a relation between L_1 and L_2 so that only one length scale is left.

Using (90), one finds for Eq. (89)

$$L_2^2 - 6L_1 L_2 - 3L_1^2 - 3\lambda \eta^{-1} L_1^{1/2} = 0$$

with the positive solution

$$L_2 = 3L_1[1 + (\tfrac{4}{3} + \tfrac{1}{3}\lambda\eta^{-1}L_1^{-3/2})^{1/2}] \tag{92}$$

The contribution from B is

$$\begin{aligned} &B(L_1;\lambda H_0 a^{1/2} L_1^{1/2}) \\ &= \text{Prob}\left\{\max_{0<l<L_1} H_0 a^{1/2} \sum_{\mu=0}^{l} X_\mu < H_0 a^{1/2}\lambda L_1^{1/2}\right\} \\ &= 1 - \text{Prob}\left\{\max_{0<l<L_1} \sum_{\mu=0}^{l} X_\mu \geqslant \lambda L_1^{1/2}\right\} \\ &\geqslant 1 - 2\exp\left(-\frac{1}{2L_1}\lambda^2 L_1\right) \qquad \text{(using Section 6.4)} \end{aligned} \tag{93}$$

Relation (93) yields a lower bound for λ since

$$B > 0, \qquad \text{i.e.,} \qquad \lambda > (2\ln 2)^{1/2} \simeq 1.18 \tag{94}$$

To get (92) independent of any parameter, the choice for the remaining length scale

$$\eta L_2^{3/2} = 1.0333 \tag{95}$$

is implied. Choosing a certain λ, L_1 is determined as

$$L_1 = 0.04 L_2 \qquad \text{for} \quad \lambda = 1.3 \tag{96}$$

i.e.,

$$B > 1 - 2e^{-0.85} = 0.14 \tag{97}$$

The theory of random walks implies

$$\int_0^\infty dv\, Y(L_2 - L_1; v + 2H_0 a^{1/2}\eta L_2^2) \cong 0.0174 \tag{98}$$

The contribution from the first interval reads

$$\begin{aligned} &\int_0^\infty du\, A_0(L_1; \tfrac{1}{3}a^{1/2}H_0 L_2^2, u) \\ &= \text{Prob}\left\{\min_{0\leqslant l<L_1} H_0 a^{1/2}\sum_{\mu=0}^{l} X_\mu > -\tfrac{1}{3}a^{1/2}H_0\eta L_2^2 \text{ and } \sum_{\mu=0}^{L_1} X_\mu \leqslant 0\right\} \\ &= \text{Prob}\left\{\min_{0\leqslant l<L_1} \sum_{\mu=0}^{l} X_\mu > -1.72 L_1^{1/2} \text{ and } \sum_{\mu=0}^{L_1} X_\mu \leqslant 0\right\} \end{aligned}$$

Splitting $\sum_{\mu=0}^{L_1} X_\mu \leqslant 0$ into disjoint events

$$\text{Prob}\left\{\sum_{\mu=0}^{L_1} X_\mu \leqslant 0\right\}$$

$$= \text{Prob}\left\{\sum_{\mu=0}^{L_1} X_\mu \leqslant 0 \text{ and } \min_{0 \leqslant l < L_1} \sum_{\mu=0}^{l} X_\mu > -1.72 L_1^{1/2}\right\}$$

$$+ \text{Prob}\left\{\sum_{\mu=0}^{L_1} X_\mu \leqslant 0 \text{ and } \min_{0 \leqslant l < L_1} \sum_{\mu=0}^{l} X_\mu \leqslant -1.72 L_1^{1/2}\right\}$$

implies

$$\text{Prob}\left\{\sum_{\mu=0}^{L_1} X_\mu \leqslant 0 \text{ and } \min_{0 \leqslant l < L_1} \sum_{\mu=0}^{l} X_\mu > -1.72 L_1^{1/2}\right\}$$

$$\geqslant \tfrac{1}{2} - \text{Prob}\left\{\min_{0 \leqslant l < L_1} \sum_{\mu=0}^{l} X_\mu > -1.72 L_1^{1/2}\right\}$$

$$\geqslant \tfrac{1}{2} - 2\exp[-(1/2L_1)(1.72 L_1^{1/2})^2]$$

$$\geqslant \tfrac{1}{2} - 2\exp[-(1.72^2/2)] \tag{99}$$

The inequality (97) and those following it imply for p_1

$$p_1 > 0.14 \times 0.0174 \times 0.046 > 10^{-4} \tag{100}$$

and for the correlation function

$$\overline{\langle l^2 \rangle} \gtrsim 3a^{-2}\left(\frac{H_0}{g}\right)^{4/3} \times 10^{-6} \tag{101}$$

Together with the upper bound for $\overline{\langle l^2 \rangle}$ derived in the preceding section, we have the result that for low temperature and strong random field the correlation scales like $(z = l \cdot a)$

$$\overline{\langle z^2 \rangle} \sim (H_0/g)^{4/3} \tag{102}$$

6.4. Upper Bound for Prob$\{\min_{0 \leqslant l < L} \sum_{\mu=0}^{l} X_\mu \leqslant -m\}$

In this section an upper bound for the probability is constructed that the minimum of all symmetric random walks of length L does not exceed $-m$ $(m \geqslant 0)$. Define

$$P_0(L;r) = \text{Prob}\left\{\sum_{\mu=0}^{L} X_\mu = r\right\}$$
$$= P_0(L;-r)$$
$$= 2^{-L} L! \Big/ \left[\left(\frac{L+r}{2}\right)! \left(\frac{L-r}{2}\right)!\right] \tag{103}$$

Now consider

$$Q_0(L;r,m) = \text{Prob}\left\{\min_{0 \leqslant l \leqslant L} \sum_{\mu=0}^{l} X_\mu \leqslant -m \text{ and } \sum_{\mu=0}^{L} X_\mu = r\right\}$$

with (a) $m<0$ *or* $r \leqslant -m$: in this case $Q_0(L;r,m) = P_0(L;r)$, since the conditions do not represent restrictions to the random walks; or (b) $m \geqslant 0$ *and* $r > -m$: to every random walk with endpoint (L, r) and minimum $\leqslant -m$ there is exactly a corresponding one starting at $-2m$ and reaching the same endpoint: a random walk satisfying (b) hits the $(-m)$ line at least once, say at $l = l_0$ for the first time. Reflect the first part of this random walk as indicated in Fig. 4 and one ends up with a random walk traveling the distance $2m + r$ in L steps *without* any further restriction. This construction is unique. Since the starting value of the random walk is irrelevant to its probability, one gets

(b) $Q_0(L;r,m) = P_0(L;2m+r)$

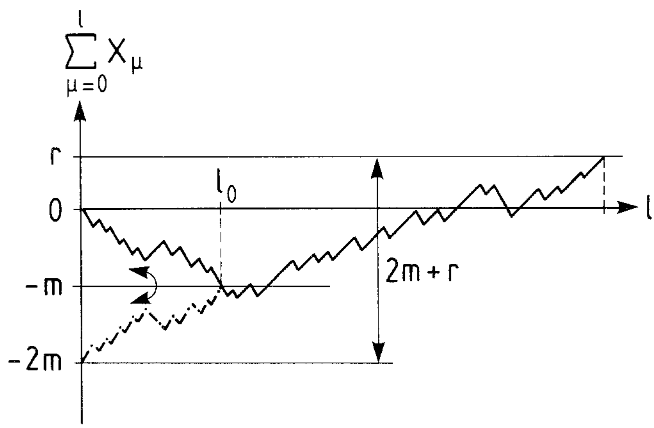

Fig. 4. Random walk with $\min_{0 \leqslant l < L} \sum_{\mu=0}^{l} X_\mu \leqslant -m$ and $\sum_{\mu=0}^{L} X_\mu = r$.

Because (a) and (b) are disjoint events,

$$\begin{aligned}\text{Prob}\left\{\min_{0\leqslant l\leqslant L}\sum_{\mu=0}^{l} X_\mu \leqslant -m\right\} &= \sum_{r=-L}^{+L} Q_0(L; r, m)\\ &= \sum_{r\leqslant -m} P_0(L; r) + \sum_{r>-m} P_0(L; 2m+r)\\ &\leqslant 2\sum_{r\geqslant 0} P_0(L; m+r) \end{aligned}\tag{104}$$

Using Stirling's formula, we obtain

$$\begin{aligned}&\text{Prob}\left\{\min_{0\leqslant l\leqslant L}\sum_{\mu=0}^{l} X_\mu \leqslant -m\right\}\\ &\leqslant 2\left(\pi\frac{L}{2}\right)^{-1/2}\sum_{r\geqslant 0}\exp\left(-\frac{(m+r)^2}{2L}\right)\\ &\leqslant 2\pi^{-1/2}\exp\left(-\frac{m^2}{2L}\right)\int_0^\infty dr\,\exp\left(-\frac{r^2}{4}\right)\\ &\leqslant 2\exp\left(-\frac{m^2}{2L}\right)\end{aligned}\tag{105}$$

since m, $r\geqslant 0$ and provided that $L \gg 1$.

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NOTE ADDED IN PROOF

The inequalities derived in Section 6 are sufficient to prove (102) and thus close a controversy which has appeared even in the case of the $(d=0)$ toy-model of Section 4. However these inequalities are too loose to give a useful information for numerical calculations. The prefactor of the right hand side of (105) is certainly too generous. On the other hand, the lower bound (101) can be improved if $-2E$ is replaced by $-E$ in the definition (79) of p_1.

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